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Curvature-Based Computation of Antipodal Grasps*

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Abstract

It is well known that antipodal grasps can be achieved on curved objects in the presence of friction. This paper presents an efficient algorithm that finds, up to numerical resolution, all pairs of antipodal points on a closed, simple, and twice continuously differentiable plane curve. Dissecting the curve into segments everywhere convex or everywhere concave, the algorithm marches simultaneously on a pair of such segments with provable convergence and interleaves marching with numerical bisection. It makes use of new insights into the differential geometry at two antipodal points. We have avoided resorting to traditional nonlinear programming which would neither be quite as efficient nor guarantee to find all antipodal points. Dissection and the coupling of marching with bisection introduced in this paper are potentially applicable to many optimization problems involving curves and curved shapes.

1 Introduction

A grasp on an object is *force closure* if and only if arbitrary force and torque can be exerted on the object through the finger contacts. Two fingers in frictional contact with a 2D curved shape can form a force-closure grasp if they are placed at two points whose inward normals are opposite and collinear. Such a grasp is referred to as an *antipodal grasp* while the two points are referred to as *antipodal points*. For example, the closed curve in Figure 1 has eight pairs of antipodal points (numbered the same within each pair).

We present a fast algorithm that finds all antipodal points on a general closed plane curve. The novelty of this algorithm lies in (a) its dissection of the curve into segments turning in one direction, and (b) its combination of numerical bisection and marching with provable convergence.

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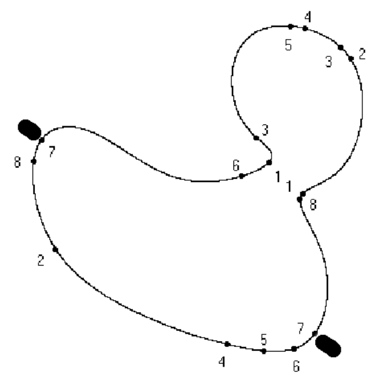


Figure 1: Eight pairs of antipodal points on a curved shape. The grasp at pair 7 (or at any other pair) is force-closure.

1.1 Related Work

Hong *et al.* [5] proved the existence of two pairs of antipodal points on a closed, simple, and smooth convex curve or surface. Chen and Burdick [3] computed antipodal points on 2D and 3D shapes through minimizing a grasping energy function. Blake and Taylor [2] gave a geometric classification of two-fingered frictional grasps of smooth contours. Ponce *et al.* [11] employed parallel cell decomposition to compute pairs of maximal-length segments on a piecewise-smooth curved 2D object that guarantee force closure with friction.

Nguyen [9] described simple algorithms for synthesizing independent grasp regions on polygons and polyhedra, with or without friction. Markenscoff *et al.* [8] determined the number of fingers to immobilize 2-D and 3-D objects with piecewise smooth boundaries. We refer the reader to [1] for a survey of research on grasping and contact.

In preprocessing our algorithm finds points of simple inflection. In [4] Goodman gave an upper bound on the number of inflection points on parametric spline curves. Manocha and Canny [7] used the Sturm sequence method to find inflection points on rational curves. Sakai [12] obtained the distribution of inflection points and cusps on a parametric rational cubic curve.

1.2 Antipodal Points

Let $\boldsymbol{\alpha}(u)$ be a closed, simple, and twice continuously differentiable curve, where u increases counterclockwise. For clarity of presentation, we assume that $\boldsymbol{\alpha}$ is *unit-speed*, that is, $\|\boldsymbol{\alpha}'(u)\| = 1$. All procedures in this paper are presented on unit-speed curves but can be extended with virtually no effort to (and have been implemented on) arbitrary-speed curves.

Denote by $T(u) = \boldsymbol{\alpha}'(u)$ the tangent of $\boldsymbol{\alpha}$ and denote by $N(u)$ the inward normal. We only consider that $\boldsymbol{\alpha}$'s curvature κ is not constant.[1] Furthermore, κ can be zero at only isolated points on the curve. In case no ambiguity arises, the parameter u also refers to the point $\boldsymbol{\alpha}(u)$ on the curve. Two points a and b on $\boldsymbol{\alpha}$ are called *antipodal* if their normals are opposite and collinear:

$$N(a) + N(b) = 0 \quad \text{and} \quad N(a) \times \Big(\boldsymbol{\alpha}(b) - \boldsymbol{\alpha}(a)\Big) = 0.$$

In Section 2 we will consider how to find antipodal points on a pair of segments of $\boldsymbol{\alpha}$ that satisfy some restricted conditions. In Section 3 we will describe how to preprocess $\boldsymbol{\alpha}$ to generate all such pairs. Section 4 will present some experimental results.

2 Computation of Antipodal Points

Two segments of $\boldsymbol{\alpha}$, denoted as $\mathcal{S}$ and $\mathcal{T}$, are defined on subdomains (s_a, s_b) and (t_a, t_b), respectively. Here $s_a < s_b$ always holds. For convenience, we allow $t_a > t_b$, in which case (t_a, t_b) refers to the interval (t_b, t_a). Let $\Phi(a, b) = \int_a^b \kappa \, du$ be the *total curvature* over (a, b), which measures the amount of rotation of the tangent T as it moves from a to b along the curve. We assume that the following conditions are satisfied:

(i) No intersection between $\mathcal{S}$ and $\mathcal{T}$.

(ii) $\kappa > 0$ everywhere or $\kappa < 0$ everywhere on both $\mathcal{S}$ and $\mathcal{T}$, with $\kappa = 0$ possible only at s_a, s_b, t_a, and t_b.

(iii) $N(s_a) + N(t_a) = 0$ and $N(s_b) + N(t_b) = 0$ but neither s_a and t_a nor s_b and t_b are antipodal.

(iv) $-\pi \le \Phi(s_a, s_b) = -\Phi(t_a, t_b) \le \pi$.

Conditions (ii) states that the normal rotates in one direction as each segment is traversed. Condition (iv) ensures that a pair of antipodal points cannot appear on the same segment, which does not include s_a, s_b, t_a, or t_b.

Under condition (iii) (and (ii) and (iv)), a one-to-one correspondence exists between a point s on $\mathcal{S}$ and a point t on $\mathcal{T}$:

$$\begin{aligned} N(s) + N(t) &= 0, \qquad \text{or equivalently,} && (1)\\ T(s) + T(t) &= 0. && (2) \end{aligned}$$

Let $g(s,t) = N(s) \times N(t)$. Since $\frac{\partial g}{\partial t} = N(s) \times (-\kappa(t)T(t)) = -\kappa(t) \neq 0$,[2] by the Implicit Function Theorem, the equation $g(s,t) = 0$ defines t as a function of s. We refer to t as the *opposite point* of s.

A pair of points may be antipodal only if their normals do not point away from each other. We add a fifth condition:

(v) $N(s) \cdot \Big(\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\Big) > 0$ for all $s \in (s_a, s_b)$.

Differentiate (2) and then plug (1) in: $(\kappa(s) - \kappa(t)\frac{dt}{ds})N(s) = 0$. Thus $\kappa(s) - \kappa(t)\ \frac{dt}{ds} = 0$ and $\frac{dt}{ds} = \frac{\kappa(s)}{\kappa(t)}$.

2.1 Antipodal Angle

Define the *antipodal angle*[3] $\theta(s)$ as the rotation angle from the normal $N(s)$ to the vector $\boldsymbol{r}(s) = \boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)$ (see Figure 2). Under condition (v), $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. By definition, s and t are antipodal *if and only if* $\theta(s) = 0$.

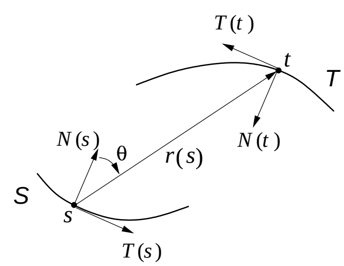


Figure 2: Antipodal angle θ.

To determine θ', we first calculate the derivative:

$$\frac{d}{ds}\|\boldsymbol{r}(s)\| = \frac{\big(T(t)\frac{dt}{ds} - T(s)\big) \cdot \boldsymbol{r}(s)}{\|\boldsymbol{r}(s)\|} = \left(\frac{\kappa(s)}{\kappa(t)} + 1\right)\sin\theta.$$

From Figure 2 we see that $\sin\theta = N(s) \times \boldsymbol{r}(s)/\|\boldsymbol{r}(s)\|$. Differentiate both sides of this equation and substitute $\frac{d}{ds}\|\boldsymbol{r}(s)\|$ in. After a few more steps, we obtain

$$\theta'(s) \quad = \quad -\kappa(s) + \frac{\cos\theta}{\|\boldsymbol{r}(s)\|}\left(\frac{\kappa(s)}{\kappa(t)} + 1\right). \qquad (3)$$

Two antipodal point s^* and t^* with $\theta'(s^*) \neq 0$ are called *simple antipodal points*.

The rest of Section 2 presents an algorithm to find all simple antipodal points on $\mathcal{S}$ and $\mathcal{T}$. This algorithm deals separately with three cases: $\mathcal{S}$ and $\mathcal{T}$ are both concave, both convex, or one concave and the other convex.

[1] This excludes a circle on which any two points determining a diameter are antipodal.

[2] The Frenet formulas [10, pp. 56–58] for planar curves state that $T'(s) = \kappa(s)N(s)$ and $N'(s) = -\kappa(s)T(s)$.

[3] In [2], it is referred to as the friction angle.

2.2 Two Concave Segments

In this case, $\kappa(s) < 0$ and $\kappa(t) < 0$; by (3), $\theta'(s) > 0$. The antipodal angle θ increases monotonically from s_a to s_b.

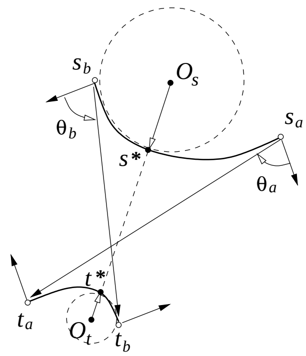


Figure 3: Two concave segments. Since $\theta(s_a) = \theta_a < 0$ and $\theta(s_b) = \theta_b > 0$ exactly one pair of antipodal points exists.

Theorem 1 *Suppose $\mathcal{S}$ and $\mathcal{T}$ are concave. If $\theta(s_a) < 0$ and $\theta(s_b) > 0$ then a unique pair of antipodal points exists. Otherwise, no antipodal points exist.*

When $\theta(s_a) < 0$ and $\theta(s_b) > 0$, we use bisection to find the antipodal points. Initialize $(s_0, t_0) \leftarrow (s_a, t_a)$ and $(s_1, t_1) \leftarrow (s_b, t_b)$. Then evaluate $s_2 \leftarrow \frac{s_0+s_1}{2}$ and find its opposite point t_2. If $\theta(s_2) > 0$, set $(s_1, t_1) \leftarrow (s_2, t_2)$; otherwise set $(s_0, t_0) \leftarrow (s_2, t_2)$. Repeat the above steps until $\theta(s_2)$ approaches 0, that is, until s_2 and t_2 approach two antipodal points.

2.3 Two Convex Segments

Since $\kappa(s) > 0$ over $\mathcal{S}$ and $\kappa(t) > 0$ over $\mathcal{T}$, we cannot determine the sign of $\theta'(s)$. Multiple pairs of antipodal points may exist on $\mathcal{S}$ and $\mathcal{T}$. The first pair will be found through "marching" described in Section 2.3.1 if $\theta(s_a)$ and $\theta(s_b)$ have the same sign or through bisection in Sections 2.3.2 if they have different signs. Section 2.3.3 will describe how all the remaining pairs can be found by letting the two strategies invoke each other recursively.

2.3.1 Endpoint Antipodal Angles with the Same Sign

The marching strategy will rely on the following result.

Proposition 2 *When $\mathcal{S}$ and $\mathcal{T}$ are convex, the vector $\boldsymbol{r}(s)$ rotates counterclockwise as s increases from s_a to s_b.*

Proof We need only show that $\frac{d\boldsymbol{r}}{ds} \times \boldsymbol{r} < 0$. Differentiating the vector $\boldsymbol{r}$ yields

$$\frac{d\boldsymbol{r}}{ds} = \frac{d}{ds}\Big(\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\Big) = T(t)\left(1 + \frac{\kappa(s)}{\kappa(t)}\right).$$

Since $\kappa(s), \kappa(t) > 0$, we have $1 + \frac{\kappa(s)}{\kappa(t)} > 0$. Hence $\frac{d\boldsymbol{r}}{ds}$ is in the direction of $T(t)$. Meanwhile, from condition (v) that $\boldsymbol{r}(s) \cdot N(s) > 0$ it follows that $T(s) \times \boldsymbol{r}(s) > 0$ and $T(t) \times \boldsymbol{r}(s) < 0$. Therefore $\frac{d\boldsymbol{r}(s)}{ds} \times \boldsymbol{r}(s) < 0$. □

Figure 4 illustrates the working of an iterative method when $\theta(s_a) < 0$ and $\theta(s_b) < 0$. The iteration starts with s and t at $s_0 = s_b$ and $t_0 = t_b$, respectively. From Proposition 2, as s moves towards s_a, the vector $\boldsymbol{r}(s)$ rotates clockwise. At the ith iteration step move s from s_i to s_{i+1} at which the normal is parallel to $\boldsymbol{r}(s_i)$. If no such point s_{i+1} exists, stop. Otherwise, move t from t_i to t_{i+1} where $N(t_{i+1}) + N(s_{i+1}) = 0$. The iteration continues until

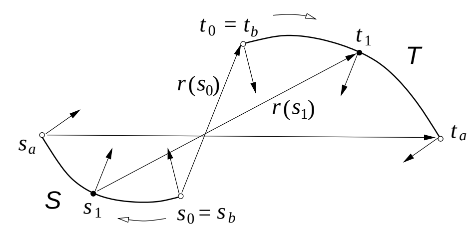


Figure 4: The working of the procedure `Antipodal-Convex-March` when $\theta(s_a) < 0$ an $\theta(s_b) < 0$.

s_i and t_i converge to a pair of antipodal points, as in Figure 5(a), or they reach s_a and t_a, in which case no antipodal points exist as in Figure 4.

When $\theta(s_a) > 0$ and $\theta(s_b) > 0$, the march starts at s_a and t_a and moves towards s_b and t_b, respectively, in the same manner. The method has been implemented in the procedure `Antipodal-Convex-March`.

Below we establish the correctness of the procedure when $\theta(s_a) < 0$ and $\theta(s_b) < 0$.

Lemma 3 *In the case $\theta(s_a) < 0$ and $\theta(s_b) < 0$ of the procedure* `Antipodal-Convex-March`, *$s_i > s_{i+1}$ and every $s \in [s_{i+1}, s_i)$ satisfies $\theta(s) < 0$ for all $i \geq 0$.*

Proof We use induction. That $\theta(s_0) = \theta(s_b) < 0$ follows directly from the initial condition. Suppose $\theta(s_i) < 0$. The normal $N(s)$ rotates clockwise as s decreases from s_i. Also since $N(s_i) \times \boldsymbol{r}(s_i) < 0$ and the normal $N(s_{i+1})$, if s_{i+1} exists, is in the direction of $\boldsymbol{r}(s_i)$, we know that $s_{i+1} < s_i$ and

$$N(s) \times \boldsymbol{r}(s_i) \quad < \quad 0, \qquad \text{for all } s \in (s_{i+1}, s_i). \quad (4)$$

By Proposition 2, $\boldsymbol{r}(s_i)$ rotates clockwise as s moves from s_i to s_{i+1}; hence

$$\boldsymbol{r}(s_i) \times \boldsymbol{r}(s) < 0, \qquad \text{for all } s \in [s_{i+1}, s_i). \quad (5)$$

Combining inequalities (4) and (5) with condition (v) that $N(s) \cdot \boldsymbol{r}(s) > 0$ over (s_a, s_b), we infer that

$$N(s) \times \boldsymbol{r}(s) < 0, \qquad \text{for all } s \in [s_{i+1}, s_i).$$

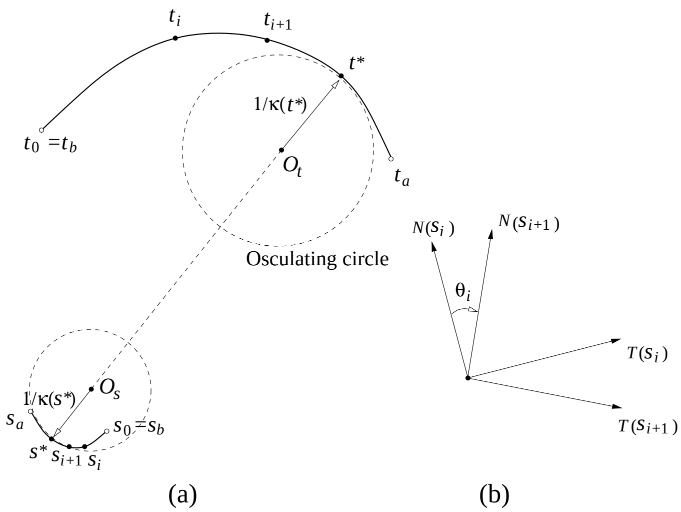


Figure 5: Geometry at the antipodal points s^* and t^*: $\theta'(s^*) < 0$ and s^* is closer to the center of curvature O_s at s^* than to the center of curvature O_t at t^*. Here $\theta_i = \theta(s_i)$.

Thus $\theta(s) < 0$ for all $s \in [s_{i+1}, s_i)$. □

Lemma 3 states that the sequence $s_0, s_1, \ldots$, defined by

$$N(s_{i+1}) \times \boldsymbol{r}(s_i) = 0 \tag{6}$$

is monotonically decreasing and no antipodal point exists on $[s_i, s_b) = \cup_{k=1}^{i} [s_k, s_{k-1})$ for all $i > 0$. Suppose the segment $\mathcal{S}$ has at least one antipodal point and let s^* be the one closest to s_b. See Figure 5(a). Then the monotonic sequence $\{s_i\}$ is bounded below by s^*. So it must converge to some $\xi \in (s_a, s_b)$ where $N(\xi) \times \boldsymbol{r}(\xi) = 0$. Hence $\xi = s^*$.

Next, we determine the local convergence rate of the sequence. Apply Taylor's expansion of the iteration function f where $s_{i+1} = f(s_i)$ defined implicitly by (6) at s^*:

$$s_{i+1} - s^* \;=\; f(s_i) - f(s^*) \;=\; f'(s^*)(s_i - s^*) + \cdots.$$

Below we determine $f'(s^*)$. For simplicity, denote the antipodal angle $\theta(s_i)$ by θ_i. As shown in Figure 5(b), $\sin\theta_i = N(s_i) \times N(s_{i+1})$. Differentiating both sides of this equation with respect to s_i yields

$$\begin{aligned} \cos\theta_i\, \theta'(s_i) &= -\kappa(s_i) T(s_i) \times N(s_{i+1}) + \\ &\quad N(s_i) \times \Big(-\kappa(s_{i+1}) f'(s_i) T(s_{i+1})\Big) \\ &= -\kappa(s_i)\cos\theta_i + \kappa(s_{i+1}) f'(s_i)\cos\theta_i. \end{aligned}$$

Let t^* be the opposite point of s^*. Hence we have

$$f'(s^*) = \frac{\theta'(s^*)}{\kappa(s^*)} + 1 = \frac{\kappa(s^*) + \kappa(t^*)}{\kappa(s^*)\kappa(t^*)\|\boldsymbol{r}(s^*)\|} > 0. \tag{7}$$

Note that the iteration starts at s_b where $\theta(s_b) < 0$ and never passes s^*. So $\theta'(s^*) < 0$ must hold in the non-degenerate case. This and that $\kappa(s^*) > 0$ imply that $f'(s^*) < 1$. Therefore the convergence rate is linear.

The correctness and linear convergence rate for the case $\theta(s_a) > 0$ and $\theta(s_b) > 0$ can be established similarly.

Theorem 4 *Let $\mathcal{S}$ and $\mathcal{T}$ both be convex. Suppose the two antipodal angles $\theta(s_a)$ and $\theta(s_b)$ have the same sign. Then the following statements hold:*

1. *When no antipodal points exist on $\mathcal{S}$ and $\mathcal{T}$, the procedure* `Antipodal-Convex-March` *terminates at s_a and t_a if $\theta(s_a) < 0$ and $\theta(s_b) < 0$ or at s_b and t_b if $\theta(s_a) > 0$ and $\theta(s_b) > 0$.*

2. *Otherwise, the procedure converges at linear rate to a pair of antipodal points s^* and t^* closest to the two endpoints at which the iteration starts. Furthermore, $\theta'(s^*) < 0$ must hold.*

2.3.2 Endpoint Antipodal Angles with Different Signs

In this case, the two antipodal angles $\theta(s_a)$ and $\theta(s_b)$ have different signs. At least one pair of antipodal points exists. To find one pair, we use a bisection procedure `Antipodal-Convex-Bisect`. At the found antipodal points s^* and t^*, either $\theta'(s^*) > 0$ or $\theta'(s^*) < 0$.

2.3.3 Finding All Pairs of Antipodal Points

After finding one pair of antipodal points s^* and t^*, how do we move on to find other pairs of antipodal points if they exist on convex segments $\mathcal{S}$ and $\mathcal{T}$?

Suppose $\theta(s_a)$ and $\theta(s_b)$ have the same sign. Then s^* and t^* have been found by the procedure `Antipodal-Convex-March`. Let us consider the case that $\theta(s_a) < 0$ and $\theta(s_b) < 0$. No antipodal points exist in (s^*, s_b) since the iteration started at s_b and ended at s^*. That $\theta'(s^*) < 0$ and $\theta(s^*) = 0$ imply $\theta(s^* - \epsilon) > 0$ for small enough $\epsilon > 0$. Therefore the interval $(s_a, s^* - \epsilon)$ contains at least one antipodal point. So we need to invoke the procedure `Antipodal-Convex-Bisect`$(s_a, s^* - \epsilon, t_a, t^* - \delta)$, where $t^* - \delta$ is the opposite point[4] of $s^* - \epsilon$. Similarly, when $\theta(s_a) > 0$ and $\theta(s_b) > 0$, the interval $(s^* + \epsilon, s_b)$ contains at least one antipodal point. We need to invoke `Antipodal-Convex-Bisect`$(s^* + \epsilon, s_b, t^* + \delta, t_b)$.

Suppose $\theta(s_a)$ and $\theta(s_b)$ have different signs. Then s^* and t^* are found by `Antipodal-Convex-Bisect`. And $\theta(s^* - \epsilon)$ has the sign of $\theta(s_a)$ while $\theta(s^* + \epsilon)$ has the sign of $\theta(s_b)$. The procedure `Antipodal-Convex-March` needs to be invoked on both intervals $(s_a, s^* - \epsilon)$ and $(s^* + \epsilon, s_b)$ to search for possible antipodal points.

Figure 6 illustrates the above procedure on an ellipse.

[4] Note that $\delta > 0$ if $t_a < t_b$ and $\delta < 0$ otherwise.

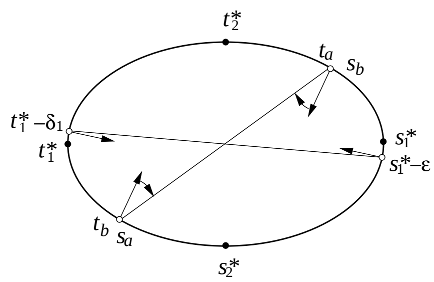


Figure 6: Two pairs of antipodal points on a pair of convex elliptic segments over (s_a, s_b) and (t_a, t_b), where $\boldsymbol{\alpha}(s_a) = \boldsymbol{\alpha}(t_b)$, $\boldsymbol{\alpha}(s_b) = \boldsymbol{\alpha}(t_a)$, and $N(s_a) = -N(t_a)$. `Antipodal-Convex-March` starts the iteration at s_b and t_b and finds the first pair of antipodal points s_1^* and t_1^*. Next, `Antipodal-Convex-Bisect` is invoked on $(s_a, s_1^* - \epsilon)$ and $(t_a, t_1^* - \delta_1)$, where $N(t_1^* - \delta_1) = -N(s_1^* - \epsilon)$, and finds a second antipodal pair s_2^* and t_2^*. Finally, `Antipodal-Convex-March` is invoked again, on the pair $(s_a, s_2^* - \epsilon)$ and $(t_a, t_2^* - \delta_2)$ and on the pair $(s_2^* + \epsilon, s_1^* - \epsilon)$ and $(t_2^* + \delta_3, t_1^* - \delta_1)$, respectively. It finds no more antipodal points.

2.4 Convex and Concave Segments

Without loss of generality, suppose $\mathcal{S}$ is convex and $\mathcal{T}$ is concave. We again compare the signs of the antipodal angles at the two endpoints of $\mathcal{S}$.

2.4.1 Endpoint Antipodal Angles with the Same Sign

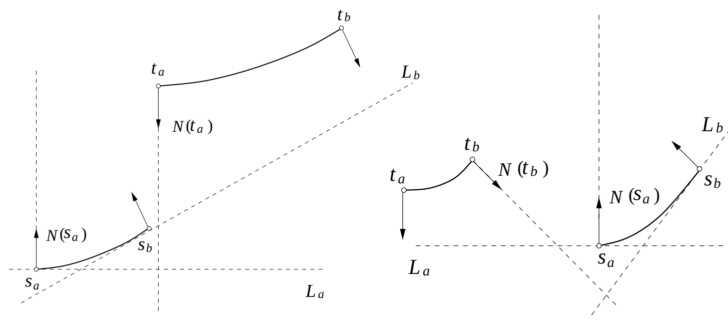
We first determine if one of the rays extending the normals $N(t_a)$ and $N(t_b)$ intersects $\mathcal{S}$. Under conditions (i)–(v), testing if the ray extending $N(t_a)$, or simply called the *ray of* $N(t_a)$, intersects $\mathcal{S}$ can be done by checking whether the cross products $(\boldsymbol{\alpha}(s_a) - \boldsymbol{\alpha}(t_a)) \times N(t_a)$ and $(\boldsymbol{\alpha}(s_b) - \boldsymbol{\alpha}(t_a)) \times N(t_a)$ have different signs.

Proposition 5 *Suppose $\mathcal{S}$ is convex and $\mathcal{T}$ is concave. Assume that the two antipodal angles $\theta(s_a)$ and $\theta(s_b)$ have the same sign. No antipodal points exist on $\mathcal{S}$ and $\mathcal{T}$ if neither the ray of $N(t_a)$ nor the ray of $N(t_b)$ intersects $\mathcal{S}$.*

Proof For simplicity, we assume that $N(s_a)$ points vertically upward, as shown in Figure 7. Under condition (v), $\mathcal{S}$ and $\mathcal{T}$ must lie on the same side of the two tangent lines L_a and L_b of $\mathcal{S}$ at s_a and s_b, respectively.

Suppose neither of the rays of $N(t_a)$ and $N(t_b)$ intersects $\mathcal{S}$. Because $N(t_a)$ does not intersect $\mathcal{S}$, t_a is either to the right of $\mathcal{S}$ or to its left. If t_a is to the right, condition (v) determines that the segment $\mathcal{T}$ cannot cross the line containing $N(t_a)$ to its left. So $\mathcal{T}$ lies entirely to the right of the segment $\mathcal{S}$, as shown in Figure 7(a). But all normals on $\mathcal{S}$ point to the left. Thus no antipodal points exist.

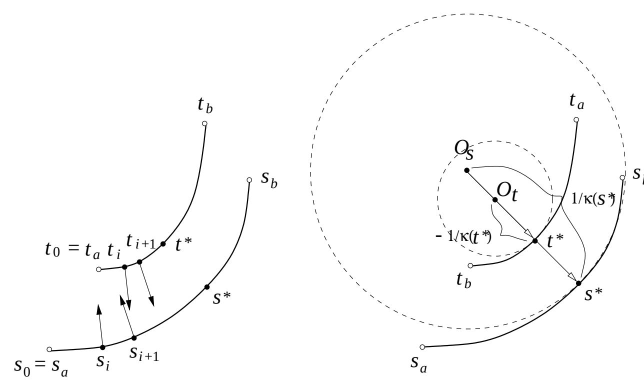
If t_a is to the left of $\mathcal{S}$, then $\theta(s_a) > 0$. Since $\theta(s_b)$ has the same sign, $\theta(s_b) > 0$. Then $\mathcal{S}$ and $\mathcal{T}$ must lie on different sides of the line containing $N(t_b)$ as in Figure 7(b). Apparently, they cannot have antipodal points either. □



(a) (b)

Figure 7: No antipodal points. Neither the ray of $N(t_a)$ nor the ray of $N(t_b)$ intersects the segment $\mathcal{S}$.

The iteration starts at $s_0 = s_a$ and $t_0 = t_a$ if the ray of $N(t_a)$ intersects $\mathcal{S}$, or at $s_0 = s_b$ and $t_0 = t_b$ if the ray of $N(t_b)$ intersects $\mathcal{S}$. In each round, s_{i+1} is generated as the intersection of the ray of $N(t_i)$ and $\mathcal{S}$ and t_{i+1} is generated as its opposite point. The iteration stops if the sequences $s_0, s_1, \ldots$ and $t_0, t_1, \ldots$ reach the other endpoints, in which case no antipodal points exist, or if they converge to a pair of antipodal points. Figure 8(a) illustrates this iteration when the ray of $N(t_a)$ intersects $\mathcal{S}$.



(a) (b)

Figure 8: The iteration when the ray of $N(t_a)$ intersects $\mathcal{S}$.

Lemma 6 *Suppose $\mathcal{S}$ is convex and $\mathcal{T}$ is concave. And suppose the ray of $N(t_a)$ intersects $\mathcal{S}$. In the above iteration, $s_i < s_{i+1}$ and no antipodal points exist in $(s_i, s_{i+1}]$ and $(t_i, t_{i+1}]$ for all $i \geq 0$.*

The proof of the lemma is by induction in a way similar to the proof of Lemma 3. Following Lemma 6, the sequence $\{s_i\}$ defined by

$$\Big(\boldsymbol{\alpha}(t_i) - \boldsymbol{\alpha}(s_{i+1})\Big) \times N(s_i) = 0 \tag{8}$$

is monotonically increasing. If there exists at least one antipodal point on $\mathcal{S}$, the sequence $\{s_i\}$ will converge to the first such point s^* from s_a.

To study local convergence rate of the procedure we differentiate equation (8) to obtain the derivative of the iteration function g where $s_{i+1} = g(s_i)$ at s^* [6]:

$$g'(s^*) = \kappa(s^*)\|\boldsymbol{\alpha}(t^*) - \boldsymbol{\alpha}(s^*)\| - \frac{\kappa(s^*)}{\kappa(t^*)}. \quad (9)$$

Because $\kappa(s^*) > 0$ and $\kappa(t^*) < 0$, $g'(s^*) > 0$. Because $\theta(s_i) < 0$, for $i = 0, 1, \ldots$, we see that

$$\theta'(s^*) = -\kappa(s^*) + \frac{1}{\|\boldsymbol{r}(s^*)\|}\left(\frac{\kappa(s^*)}{\kappa(t^*)} + 1\right) > 0. \quad (10)$$

in the non-degenerate case. This implies that $0 < g'(s^*) < 1$. Hence the algorithm converges in linear rate. It also follows from (10) that $\|\boldsymbol{\alpha}(t^*) - \boldsymbol{\alpha}(s^*)\| < \frac{1}{\kappa(s^*)} + \frac{1}{\kappa(t^*)}$. Geometrically, the osculating circle at s^* contains the osculating circle at t^* in its interior, as shown in Figure 8(b).

Similar analysis can be performed for the case that the ray of $N(t_b)$ intersects $\mathcal{S}$. The convergence rate is still linear and $\theta'(s^*) > 0$ also holds.

If no antipodal points exist on $\mathcal{S}$ and $\mathcal{T}$, `Antipodal-Convex-Concave-March` will terminate at the other endpoints of $\mathcal{S}$ and $\mathcal{T}$.

2.4.2 Finding All Pairs of Antipodal Points

When the antipodal angles at the two endpoints have different signs, we can use bisection to find one pair of antipodal points.

To find all pairs of antipodal points on $\mathcal{S}$ and $\mathcal{T}$, the marching procedure in Section 2.4.1 and bisection above need to recursively call each other. This is similar to the case that $\mathcal{S}$ and $\mathcal{T}$ are convex in Section 2.3.3.

3 Curve Preprocessing

The preprocessing of the curve $\boldsymbol{\alpha}$ generates all pairs of segments that satisfy conditions (i)–(v) in Section 2.1. It consists of the following four steps:

1. Compute all points of *simple inflection* on $\boldsymbol{\alpha}$. A point s is simple inflection if $\kappa(s) = 0$ but $\kappa'(s) \neq 0$. In the example in Figure 9(a), there are four inflection points z_1, z_2, z_3, and z_4. They divide $\boldsymbol{\alpha}$ into segments on which the curvature does not change sign in the interior.

2. Split every segment with total curvature beyond $[-\pi, \pi]$. In Figure 9(b), the segments over $[z_2, z_3]$ and $[z_4, z_1]$ split at the points w_1 and w_2, respectively.

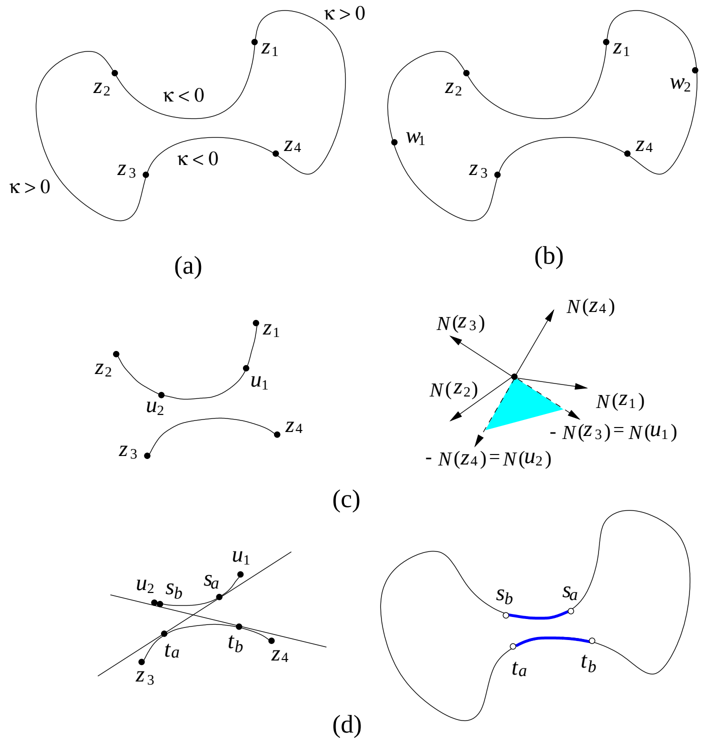


Figure 9: Four preprocessing steps.

3. Enumerate all pairs of segments. For each pair, determine if any of their endpoints are antipodal. If not, shorten the segments until condition (iii) is satisfied. To illustrate, for the pair in Figure 9(c), we intersect the cone of inward normals over $[z_1, z_2]$ with the cone of outward normals over $[z_3, z_4]$. The intersection cone (shaded) is determined by the outward normals at z_3 and z_4, which have opposite points u_1 and u_2, respectively. Accordingly, the segment $[z_1, z_2]$ is replaced with the segment $[u_1, u_2]$.

4. Now each pair satisfies conditions (i)–(iv) in Section 2.1 but not necessarily condition (v). In Figure 9(d), condition (v) is violated at both u_1 and u_2. We extract portions of the two segments divided by the tangency points s_a, t_a, s_b, and t_b of their common tangent lines. Only the portions over (s_a, s_b) and (t_a, t_b) satisfy conditions (i)–(v).

In [6] we describe an involved algorithm with quadratic convergence rate that computes common tangent lines of two curve segments.

4 Implementation

We have implemented the algorithm in C++ for arbitrary-speed curves. For details of implementation, we refer the

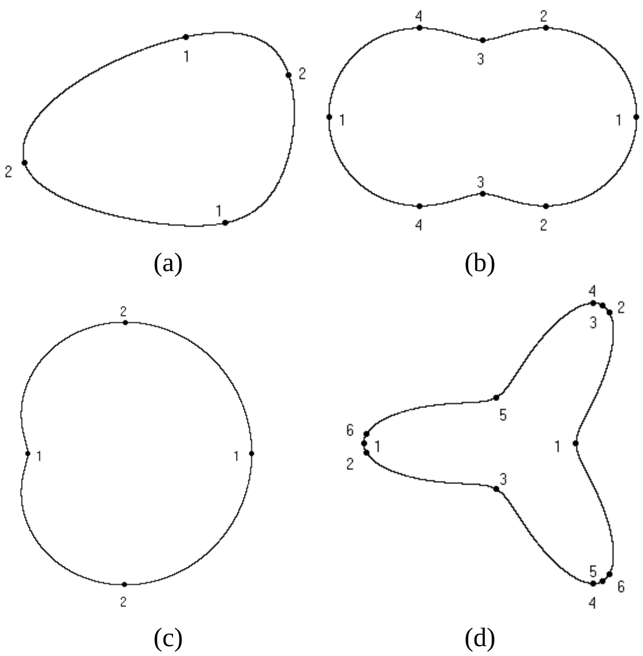


Figure 10: Antipodal points on four different shapes: (a) a convex cubic spline; (b) an elliptic lemniscate given by $\rho = \sqrt{6^2 \cos^2 \phi + 3^2 \sin^2 \phi}$ in polar coordinates; (c) a limaçon $\rho = 4 + \frac{5}{2} \cos \phi$; and (d) a curve with convexities $\rho = 3/(1 + \frac{1}{2} \cos 3\phi)$. A non-degenerate closed convex curve, like the cubic spline in (a), has two pairs of antipodal points [6]. Antipodal points in (b), (c), and (d) can be verified.

reader to [6].

The eight pairs of antipodal points in Figure 1 has all three combinations of curvature signs. Figure 10 displays all antipodal points found on four more different shapes. The first three examples each took time 10 ms on a DELL Dimension PC with Pentium III 933 MHz CPU. The fourth example took 30 ms.

Let n be the number of inflection points and m the number of pairs of antipodal points. There are $O(n^2)$ pairs of segments after the preprocessing. The total number of calls to the marching and bisection procedures in Sections 2.2–2.4 is $O(n^2 + m)$.

5 Conclusion

The algorithm described in this paper computes all antipodal points on a closed simple curve up to numerical precision. Inflection points divide the curve into segments that are either convex everywhere or concave everywhere. Such monotonicity allows a recursive combination of marching with bisection to find all antipodal points of different local geometry.

The algorithm is also applicable to a curve that is not closed, as long as an inward normal field is specified on the curve. It can also be extended in a straightforward way to a curve that is *piecewise* twice continuously differentiable.

Due to the nonlinear nature of curves, a conventional nonlinear programming approach, inherently local, would rely heavily on initial guesses of antipodal positions. It would be slow and not guarantee to always find antipodal points, not to mention all of them.

The described work will be implemented as part of our ongoing research on localization and grasping of curved objects. Future work will also include an extension of the algorithm to curved shapes in 3D.

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